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SHOCK WAVES IN AN ELASTIC PIPELINE

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The formation of shock waves from the solitary Gaussian and hyperbolic waves in an elastic pipeline has been investigated. It is shown that the parameters of these waves — the path length and the time of formation — differ insignificantly from the analogous parameters of the shock waves formed from the sine waves. The evolution of the discontinuity surface in these shock waves has been considered.

Introduction. In [1], the shock waves arising in a nonviscous liquid flowing in an elastic pipeline were investigated by the example of a sine wave. In this case, a simple wave was used as a model because the dispersion effects defined by the higher derivatives in the hydrodynamical equations are absent in this wave. At the same time, the formation of shock waves is due to the effects defined by the nonlinear terms of these equations. Therefore, we will consider shock waves within the limits of the nonlinear model of a simple wave.

The aim of the present work is to analyze the formation and development of shock waves in elastic pipelines by the example of initial conditions in the form of solitary waves obtained in [2, 3] and to determine the conditions for the minimum possible path length and time of formation of shock waves in these pipelines.

The solitary waves considered in [2, 3] represent solitons. In these waves, the nonlinear processes leading to the formation of discontinuity surfaces and the dispersion processes preventing the development of shock waves are strictly balanced.

The increase in steepness of the leading edge of pulse waves in blood vessels (Fig. 1), detected experimentally in [4], allows the conclusion that a shock wave can arise only in the idealized case of a nonviscous-liquid flow in a long elastic thin-wall pipeline. In a blood vessel, a discontinuity surface cannot be formed because of the small length of the vessel, the high viscosity of the blood, the bending moments arising in the vascular wall, and the dispersion effects occurring in a pulse wave. However, the experiments carried out in [4] point to the fact that the initial effects leading to the formation of shock waves in blood vessels manifest themselves pronouncely.

In the limit, a shock wave is formed most rapidly in the case where the dispersion processes are absent, i.e., this wave is a simple wave.

Gaussian Solitary Wave. We will consider a Gaussian solitary wave propagating in an elastic pipeline, investigated in [2]. By analogy with [1] it will be represented in the following form:

$$V = V_{\max} \exp\left[-\left(\omega\left(\tau + \frac{VX}{c^2}\right)\right)^2\right]. \qquad (1)$$

This equation determines the implicit function $V = V(X, t)$ and proposes that $V/c \ll 1$, which is true for the problem being considered.

From (1), we can determine the value of the wave phase $\omega\tau$:

$$\omega\tau = \sqrt{-\ln\left(\frac{V}{V_{\max}}\right)} - \frac{\omega VX}{c^2} = \sqrt{-\ln\left(\frac{V}{V_{\max}}\right)} - z\frac{V}{V_{\max}}. \qquad (2)$$

A coordinate system moving with a velocity c in the direction of propagation of the wave is used. Figure 2 shows the dependence of the relative longitudinal velocity of the liquid in the Gaussian wave $V/V_{\max}$ on the wave phase at the

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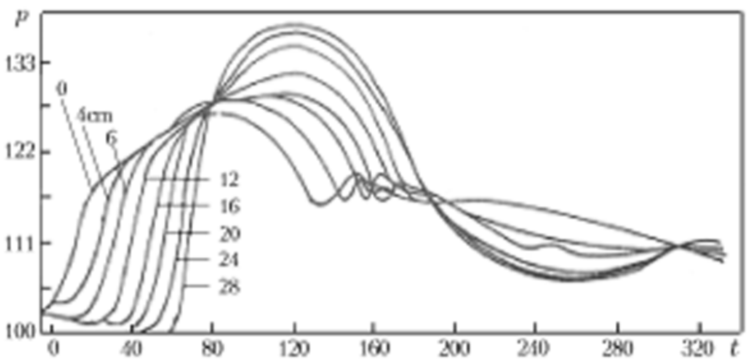

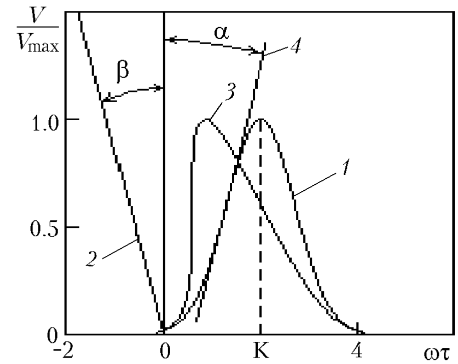

Fig. 1. Time dependence of the pressure at different sites of the aorta of a dog. Zero corresponds to the origin of the descending aorta (according to [4]). *p*, mm Hg; *t*, sec.

Fig. 2. Scheme of formation of a shock wave in an elastic pipeline for calculating the path length of its formation.

initial instant of time 1 and at the instant a discontinuity surface appears 3. Curve 1 in Fig. 2, constructed symmetrically relative to the point K by the formula $V/V_{max} = \exp\,[-(\omega\tau)^2]$, corresponds to Eq. (1) at $X = 0$. Straight line 2 was constructed by the formula $\omega\tau = -zV/V_{max}$. Curves 1 and 3 are shifted relative to the origin of coordinates by $\Delta(\omega\tau) = 2$ for better perception of the figure.

When the angle of inclination of straight line 2 ($\beta = \arctan(z)$) becomes equal to the angle of inclination α of tangent 4 to curve 1, a shock wave arises at the inflection point. Curve 3 represents the profile of the wave, the steepness of whose edge increases as the discontinuity state is approached. Since the angle of inclination of tangent 4 to curve 1 at the inflection point is equal to $\tan\left(\frac{\pi}{2} - \alpha\right) = \sqrt{2/e}$, we obtain that $z = \tan\beta = \sqrt{e/2} \approx 1.166$. In this case, the ordinate of the inflection point of curve 1 $V/V_{max} = 1/\sqrt{e}$ was used.

The condition $z = \sqrt{e/2}$ determines the minimum possible path length of formation of the shock wave:

$$X_{min} = \sqrt{\frac{e}{2}}\,\frac{c^2}{\omega V_{max}} = \frac{1}{2\pi}\sqrt{\frac{e}{2}}\,\frac{\lambda}{\mathrm{M}}. \tag{3}$$

Curve 3 in Fig. 2 was constructed on the assumption that $z = \sqrt{e/2}$. The conditions $z = \sqrt{e/2}$ and $V/V_{max} = 1/\sqrt{e}$ correspond to $\omega\tau = 0$ in Eq. (2), i.e., a discontinuity surface appears in the Gaussian wave (by analogy with the sine wave) at $\omega\tau = 0$. However, this condition does not provide the formation of shock waves from any initial waves.

We now determine the time of formation of a shock wave under the following conditions at the discontinuity surface [5]: $\partial X/\partial V = 0$ and $\partial^2 X/\partial V^2 = 0$ (Fig. 3, curve 2). Let us write the Gaussian-wave equation in form that is not related to the condition for small velocity of the liquid flow as compared to the wave velocity [1]:

$$V = V_{max}\exp\left[-\left(\omega\left(t - \frac{X}{V + c}\right)\right)^2\right]. \tag{4}$$

From [4], the coordinate X can be determined:

$$X = t\,(V + c) \pm \frac{V + c}{\omega}\sqrt{-\ln\left(\frac{V}{V_{max}}\right)}. \tag{5}$$

To the discontinuity surface corresponds the sign "+". We equate the second derivative with respect to V to zero and obtain

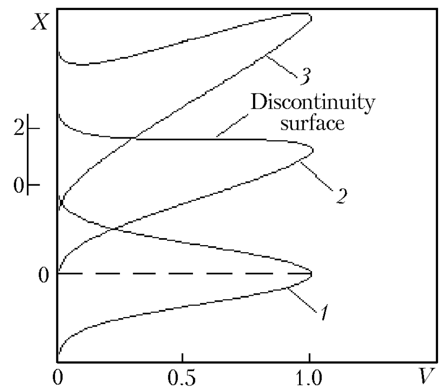

Fig. 3. Shock wave arising in an elastic pipeline. At the left, the scale of the X coordinate is given for calculating the form of the wave at different instants of time. V, m/sec.

$$\ln\left(\frac{V}{V_{\max}}\right)=\frac{1}{2}\left(\frac{V+c}{V-c}\right). \tag{6}$$

In the case where $V/c \ll 1$, the coordinate of the inflection point on the discontinuity surface will be $V/V_{\max} = 1/\sqrt{e}$. An Analogous quantity was obtained earlier for wave 1 that is symmetrical relative to the point K in Fig. 2.

The time of formation of the shock wave can be determined from the relation $\partial X/\partial V = 0$:

$$t=\frac{1}{\omega}\sqrt{-\ln\left(\frac{V}{V_{\max}}\right)}\frac{c}{V}=\frac{\lambda}{2\pi V_{\max}}\sqrt{\frac{e}{2}}\,. \tag{7}$$

A similar formula was obtained in dimensionless form in [5]. The same time can be determined also by the formula $t = X_{\min}/c$.

Hyperbolic Solitary Wave. We now perform the same analysis for the wave formed in accordance with the Korteweg and De Vries equation [3]. The initial wave has (on condition that $V/c \ll 1$) the following form:

$$V=\frac{V_{\max}}{\cosh^2\left(\omega\left(\tau+\frac{VX}{c^2}\right)\right)}\,. \tag{8}$$

We will call this wave the hyperbolic wave. The value of its phase $\omega\tau$ is determined from Eq. (8):

$$\omega\tau=\operatorname{arch}\left(1\Big/\sqrt{\frac{V}{V_{\max}}}\right)-\frac{\omega VX}{c^2}=\operatorname{arch}\left(1\Big/\sqrt{\frac{V}{V_{\max}}}\right)-z\frac{V}{V_{\max}}\,. \tag{9}$$

Calculating the ordinate at the inflection point (by analogy with the case represented by the dependence $\frac{\partial^2(\omega\tau)}{\partial(V/V_{\max})^2} = 0$ at $z = 0$, curve 1 in Fig. 2) $V/V_{\max} = 2/3$, we will determine, as in the previous case, the value of $z = \tan\beta = 3\sqrt{3}/4 \approx 1.3$. Note that the condition $\omega\tau = 0$ is not fulfilled in this case. Consequently, the path length of formation of a shock wave is equal to

$$X_{\min}=\frac{3\sqrt{3}}{4}\frac{c^2}{\omega V_{\max}}=\frac{1}{2\pi}\frac{3\sqrt{3}}{4}\frac{\lambda}{\mathrm{M}}\,. \tag{10}$$

This path length depends on the form of the initial wave giving rise to the shock wave. The path lengths of formation of a sine [1], a Gaussian, and a hyperbolic waves are related as $1:\sqrt{e/2}:3\sqrt{3}/4 \approx 1:1.166:1.3$.

If we use the hyperbolic-wave equation in a form that is not related to the condition for a small velocity of the liquid flow as compared to the wave velocity

$$V = \frac{V_{max}}{\cosh^2\left(\omega\left(t - \frac{X}{V+c}\right)\right)}, \tag{11}$$

and determine the coordinate X from (11)

$$X = t\,(V + c) - \frac{V + c}{\omega}\,\mathrm{arch}\left(1 \Big/ \sqrt{\frac{V}{V_{max}}}\right), \tag{12}$$

equating the second derivative of X with respect to V to zero will give

$$\frac{V_{max}}{V} = 1 - \frac{1}{2}\left(\frac{V + c}{V - c}\right). \tag{13}$$

On condition that $V/c \ll 1$, the coordinate of the inflection point on the discontinuity surface is $V/V_{max} = 2/3$. The identical quantity was obtained earlier for a symmetric wave. The time of formation of a shock wave can be determined from formula (10):

$$t = \frac{X_{min}}{c} = \frac{3\sqrt{3}}{4}\,\frac{c}{\omega V_{max}} = \frac{3\sqrt{3}}{4}\,\frac{\lambda}{2\pi V_{max}}. \tag{14}$$

On condition that $\partial X/\partial V = 0$, from formula (12) we can determine the value of z at the discontinuity point of a hyperbolic wave:

$$z = \frac{1}{2}\left(1 + \frac{V}{c}\right)^2 \left(\frac{V}{V_{max}}\sqrt{1 - \frac{V}{V_{max}}}\right)^{-1} \tag{15}$$

at $V/V_{max} = 2/3$ and $V/c \ll 1$, this equation gives the value of $z = 3\sqrt{3}/4$ that was obtained earlier. At $V/c \ll 1$, we have

$$z = \frac{1}{2}\left(\frac{V}{V_{max}}\sqrt{1 - \frac{V}{V_{max}}}\right)^{-1}. \tag{16}$$

For a Gaussian wave, from formula (5) at $\partial X/\partial V = 0$ we obtain

$$z = \frac{1}{2}\left(1 + \frac{V}{c}\right)^2 \left(\frac{V}{V_{max}}\sqrt{-\ln\frac{V}{V_{max}}}\right)^{-1}. \tag{17}$$

At $V/c \ll 1$

$$z = \frac{1}{2}\left(\frac{V}{V_{max}}\sqrt{-\ln\frac{V}{V_{max}}}\right)^{-1}. \tag{18}$$

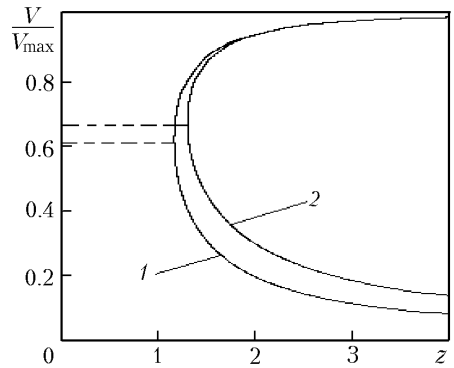

Fig. 4. Evolution of the relative longitudinal velocity of the liquid at the front of a shock wave as a function of the dimensionless length z of its propagation at $\partial X/\partial V = 0$.

We now consider the evolution of the discontinuity surface in a shock wave, i.e., the change in the form of the wave at $z \to \infty$. In [6], such an analysis was carried out for a sine wave. Figure 4 shows the dependences of the ordinates of the points V/V_{max} on the quantity z representing the longitudinal coordinate of propagation of the wave at $\partial X/\partial V = 0$. The indicated points are the points at which the tangent to the wave front is parallel to the V axis (see Fig. 3).

Curve 1 in Fig. 4, representing the Gaussian wave, was constructed by formula (18). Before the discontinuity surface appears, the ordinate of the inflection point remains constant and equal to $V/V_{max} = 1/\sqrt{e}$ at $V/c \ll 1$. Then, at the point $(\sqrt{e/2}, 1/\sqrt{e})$ there appears a discontinuity surface at which $\partial X/\partial V = 0$ and $\partial^2 X/\partial V^2 = 0$, whereupon two branches of the curve $V/V_{max} = f(z)$ appear and the total front of the wave begins to break. A further increase in z (or X) leads to the appearance of two points, at which the tangent to the wave front is parallel to the V axis in Fig. 3. Their evolution is represented by the two branches of curve 1 in Fig. 4. Curve 2 in Fig. 4, constructed by formula (16), defines a hyperbolic wave. This wave is formed at the point $3\sqrt{3}/4$, $2/3$, i.e., after the discontinuity surface appears.

The wave breaking, related to the ambiguity of the function $V/V_{max} = f(z)$, is hardly probable in the problem being considered on a liquid flow in an elastic pipeline, even thought, e.g., Eq. (5) allows one to calculate a breaking wave of the form represented by curve 3 in Fig. 3. The graphs presented in this figure were calculated for a Gaussian wave having the following parameters: $c = 8$ m/sec, $V_{max} = 1$ m/sec, $\omega = 2\pi$ sec^{-1}. Curve 1 was constructed for the initial instant of time $t = 0$, curve 2 was constructed for the moment of appearance of a discontinuity surface $t = \frac{\lambda}{2\pi V_{max}}\sqrt{\frac{e}{2}} \approx 1.484$ sec, and curve 3 was constructed for the time $t = 4$ sec.

In actuality, evidently, after a discontinuity surface is formed, the wave reflected from this surface is combined with the direct wave, with the result that the primary wave is strongly attenuated [7].

Thus, the path length and the time of formation of a shock wave in an elastic pipeline are determined by the form of the primary wave propagation in the pipeline. This is explained by the fact that different transient processes occur in different waves. However, our analysis has shown that this difference does not have a strong influence, at least for the waves being considered, on the characteristics of the wave transformation. For example, the difference between the path lengths of formation of shock waves does not exceed 30% for the sine and hyperbolic waves and 16.6% for the sine and Gaussian waves. Therefore, in calculations on the formation of shock waves in elastic pipelines, parameters that are true for the sine wave can be used initially.

Relation between the Characteristics of the Shock Waves Arising in an Elastic Pipeline and the Parameters of These Pipeline and the Liquid Flowing in It. A shock wave is formed in an elastic pipeline in the case where the dimensionless complex $\omega V_{max} X_{min}/c^2 = z_{min}$, where z_{min} depends on the form of the wave propagating in the pipeline: $z_{min} = 1$ for a sine wave [1], $z_{min} = \sqrt{e/2}$ for a Gaussian wave, and $z_{min} = 3\sqrt{3}/4$ for a hyperbolic wave.

From the relation between the velocity of a shock wave propagating in an elastic pipeline and the parameters of the pipeline walls $c = \sqrt{D/\rho} = \sqrt{E\delta/\rho d}$ [2, 4] we find the path length of formation of this wave:

$$X_{\min} = \frac{z_{\min} c^2}{\omega V_{\max}} = \frac{z_{\min} D}{\omega V_{\max} \rho} = \frac{z_{\min} E\delta}{\omega V_{\max} \rho d} . \tag{19}$$

This relation connects the path length of formation of a shock wave in an elastic pipeline to the geometric and elastic characteristics of this pipeline and the density of the liquid flowing in it.

The difference between the pressure at the front of a shock wave propagating in an elastic pipeline and the pressure in the nondisturbed pipeline can be determined using Hooke's law for an elastic pipeline [1], according to which $D = \partial(PS)/dS$. On simple rearrangements with the use of the relation $D = E\delta/d$, we obtain a linear differential equation for the independent variable d:

$$\frac{dP}{d\,(d)} + \frac{2}{d} P = \frac{2E\delta}{d^2} . \tag{20}$$

It will be assumed that the thickness of the elastic wall of the pipeline decreases insignificantly in the process of propagation of a shock wave in it: δ = const. Integration of Eq. (20) at $d = d_0$ for $P = 0$ (the excess pressure in the pipeline is equal to zero) gives the pressure at the shock-wave front

$$P \approx \frac{2E\delta}{d} \varepsilon . \tag{21}$$

Thus, the pressure differential across the front of a shock wave propagating in an elastic pipeline is proportional to the relative change in the diameter of the pipeline and depends on its geometric and elastic characteristics.

CONCLUSIONS

1. In the general case, the propagation of a solitary wave in an elastic pipeline is realized in two stages: a) the formation of a shock wave and b) the shock-wave breaking that, evidently, cannot take place in an elastic pipeline.

2. The path lengths of formation of shock waves from the sine, Gaussian, and hyperbolic waves are related as $1{:}\sqrt{e/2}{:}3\sqrt{3}/4 \approx 1{:}1.166{:}1.3$.

3. After a discontinuity surface is formed, the front of a shock wave becomes ambiguous because of the wave-front breaking.

NOTATION

c, velocity of a wave, m/sec; $D = E\delta/d$, elasticity of the walls of an elastic pipeline, N/m^2; d, diameter of the pipeline, m; d_0, diameter of the nondisturbed pipeline, m; E, modulus of elasticity of the pipeline material, N/m^2; $e \approx 2.718$, dimensionless constant; $f(z) = V/V_{\max}$, relative velocity of the liquid in the pipeline; $\mathrm{M} = V_{\max}/c$, analog of the amplitude Mach number; p, blood pressure, mm Hg; P, excess pressure in a liquid flow, Pa; S, cross-section area of the pipeline, m^2; t, time, sec; V, longitudinal velocity of the liquid flow in a wave, m/sec; $V_{\max}$, maximum velocity of the liquid flow in a wave, m/sec; X, longitudinal coordinate, m; $X_{\min}$, minimum possible path length of formation of a shock wave, m; $z = \omega V_{\max} X/c^2$, dimensionless complex (dimensionless longitudinal coordinate); $z_{\min} = \omega V_{\max} X_{\min}/c^2$, dimensionless complex; α, angle of inclination of the wave front, rad; β, angle, rad; δ, thickness of the pipeline wall, m; $\varepsilon = (d - d_0)/d$, relative deformation of the pipeline diameter; λ, wavelength, m; ρ, density of the liquid, kg/m^3; $\tau = t - X/c$, phase complex, sec; ω, cyclic frequency of the wave, rad/sec. Subscripts: 0, nondisturbed value; max and min, maximum and minimum values.

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